JOURNAL OF APPROXIMATION THEORY 36, 43–54 (1982)

Asymptotic and Generating Relations for the q-Jacobi and ${}_4\phi_3$ Polynomials

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Communicated by Paul G. Nevai

Received August 10, 1981

Generating functions, explicit representations, and uniform asymptotic formulas are derived for the little q-Jacobi polynomials, the big q-Jacobi polynomials, and the ${}_4\phi_3$ polynomials.

1. INTRODUCTION

A basic hypergeometric function ${}_{r+1}\phi_r$ is defined by

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1,..., a_r; q, x \\ b_1, b_2,..., b_r \end{matrix}\right] = \sum_{k=0}^{\infty} \frac{(a_0;q)_k}{(q;q)_k} x^k \prod_{j=1}^{r} \frac{(a_j;q)_k}{(b_j;q)_k}, \tag{1.1}$$

where the q-shifted factorial $(\lambda; q)_k$ is given by

$$(\lambda;q)_0 = 1; \quad (\lambda;q)_n = \prod_{j=1}^{n} (1-\lambda q^{j-1}), \quad n = 1, 2,..., \text{ or } n = \infty. \tag{1.2}$$

For all series considered here we shall assume that q satisfies $0 < |q| < 1$.

The little q-Jacobi polynomials are [2, 5, 18]

$$\Phi_n^{\alpha,\beta}(x) = {}_2\phi_1\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}; q, qx \\ \alpha q \end{matrix}\right]. \tag{1.3}$$

They are orthogonal with respect to a purely discrete measure on the points

* Partially supported by NSF Grant MCS-8002539-01.

0021-9045/82/090043–12$02.00/0

$x=1, q, q^2, \ldots$. These polynomials are related to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ as follows:

$$\lim_{q\to 1^-} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \Phi_n^{q^\alpha,q^\beta}(x) = P_n^{(\alpha,\beta)}(1-2x).$$

In Section 2 we shall prove the generating function

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{q^{n(n-1)/2}t^n}{(\beta q;q)_n\,(q;q)_n} \Phi_n^{\alpha,\beta}(x) \\ &= \sum_{k=0}^{\infty} \frac{(x^{-1};q)_k\,(-xt)^k}{(\beta q;q)_k\,(q;q)_k} \sum_{n=0}^{\infty} \frac{q^{n^2}(-x\alpha t)^n}{(\alpha q;q)_n\,(q;q)_n}\end{aligned} \tag{1.4}$$

and establish the asymptotic formulas

$$\Phi_n^{\alpha,\beta}(x) \sim (-x)^n\, q^{-n(n-1)/2}(x^{-1};q)_\infty/(\alpha q;q)_\infty, \tag{1.5}$$

as $n\to\infty$, $x\neq 0, 1, q, q^2, \ldots$, uniformly for x, α, and β in compact sets, and

$$\Phi_n^{\alpha,\beta}(q^m) \sim q^{(n-m)(n-m-1)/2}(-\alpha q)^{n-m}\,(\beta q^{m+1};q)_\infty/(\alpha q;q)_\infty \tag{1.6}$$

as $n\to\infty$, $m=0, 1, \ldots$, uniformly for α and β in compact sets. In the above relationships $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. In proving these, we note the usefulness of the alternate basic hypergeometric representation

$$\Phi_n^{\alpha,\beta}(x) = \frac{(\beta q;q)_n\, q^{n(n-1)/2}(-\alpha q x)^n}{(\alpha q;q)_n}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{-n}/\alpha, 1/x; q, q \\ \beta q, 0 \end{matrix}\right]. \tag{1.7}$$

The generating function (1.4) extends a generating function of Bateman, see, e.g., Rainville [15, p. 256].

The ${}_4\phi_3$ polynomials [8, 9, 22]

$$W_n(x) = a^{-n}(ab;q)_n\,(ac;q)_n\,(ad,q)_n \cdot {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, az, a/z; q, q \\ ab, ac, ad \end{matrix}\right], \tag{1.8}$$

where z is either root of $z^2 - 2xz + 1 = 0$ (so that $z = e^{\pm i\theta}$ when $x = \cos\theta$) are basic analogues of the ${}_4F_3$ polynomials in [21, 22, and 23]. They are symmetric in the four parameters a, b, c, and d as can be seen from Sears' transformation (3.1). They satisfy orthogonality relations with various weight functions, some continuous with support on $[-1, 1]$, some purely discrete, some mixed, depending on the values of the parameters. In

Section 3 we derive the generating function

$$\sum_{n=0}^{\infty} \frac{W_n(x)\, t^n}{(ab;q)_n\,(dc;q)_n\,(q;q)_n} = {}_2\phi_1\left[\begin{matrix} a/z, b/z; q, zt \\ ab \end{matrix}\right] {}_2\phi_1\left[\begin{matrix} cz, dz; q, t/z \\ cd \end{matrix}\right], \tag{1.9}$$

and the following asymptotic estimates. Set

$$A(z) = (az;q)_\infty\,(bz;q)_\infty\,(cz;q)_\infty\,(dz;q)_\infty/(z^2;q)_\infty. \tag{1.10}$$

If $x \notin [-1, 1]$ and we choose $z(z^2 - 2xz + 1 = 0)$ with $|z| < 1$, then

$$W_n(x) \sim z^{-n}A(z) \tag{1.11}$$

as $n \to \infty$, uniformly for x, a, b, c, and d in compact sets. If $x \neq \pm 1$ but is inside the ellipse with foci ± 1 and vertices $\pm \frac{1}{2}[|q|^{1/2} + |q|^{-1/2}]$, then

$$W_n(x) = z^{-n}A(z) + z^nA(z^{-1}) + 0(q^{n/2}), \tag{1.12}$$

as $n \to \infty$, uniformly for x, a, b, c, and d in compact sets. In particular, (1.12) holds for x in the interval $(-1, 1)$. If we further require that q be real and that non-real parameters among a, b, c, and d occur in conjugate pairs then $W_n(x)$ is real for real x and (1.12) reduces to

$$W_n(\cos\theta) = 2\,|A(e^{i\theta})|\cos(n\theta - \phi) + 0(q^{n/2}), \tag{1.13}$$

as $n \to \infty$, $0 < \theta < \pi$, $\phi = \arg A(e^{i\theta})$.

The ${}_4\phi_3$ polynomials contain as special cases the Al-Salam–Chihara polynomials [1] and the Rogers (or continuous) q-ultraspherical polynomials [6]. Al-Salam and Chihara [1] mentioned the weight function for their polynomials only in a very special case. The full case of the Al-Salam–Chihara polynomials, when $|q| < 1$, including asymptotic and explicit formulas, generating functions, and orthogonality relations, was treated in [7]. See also [9].

Andrews and Askey [5] introduced and studied the big q-Jacobi polynomials

$$P_n(x) = P_n(x; \alpha, \beta, \gamma : q) := {}_3\phi_2\left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, x; q, q \\ \alpha q, \gamma q \end{matrix}\right]. \tag{1.14}$$

These are also q-analogues of the ordinary Jacobi polynomials since

$$\lim_{q \to 1^-} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} P_n(x; q^\alpha, q^\beta, -1 : q) = P_n^{(\alpha,\beta)}(x).$$

These polynomials are orthogonal on a finite interval with respect to a purely discrete measure. In Section 4 we first establish the alternate basic hypergeometric representation

$$P_n(x;\alpha,\beta,\gamma:q)=\frac{(-\gamma)^n q^{n(n+1)/2}}{(\gamma q;q)_n}\left(\frac{\beta x}{\gamma};q\right)_n {}_3\phi_2\left[\begin{matrix} q^{-n},\dfrac{q^{-n}}{\beta},\dfrac{\alpha q}{x};q,q \\ \alpha q,\dfrac{\gamma q^{1-n}}{\beta x}\end{matrix}\right] \tag{1.15}$$

then use it to obtain the generating function

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\gamma q;q)_n\, t^n}{(q;q)_n\,(q\beta;q)_n}P_n(x;\alpha,\beta,\gamma:q)\\ &\quad=\left(\sum_{n=0}^{\infty}\frac{(\alpha q/x;q)_n\,(tx)^n}{(\alpha q;q)_n\,(q;q)_n}\right)\left(\sum_{k=0}^{\infty}\frac{(\beta x/\gamma;q)_k\,(-\gamma t)^k\, q^{k(k+1)/2}}{(q;q)_k\,(q\beta;q)_k}\right).\end{aligned} \tag{1.16}$$

This is also a generalization of Bateman's generating function [15, p. 256]. Furthermore, we prove the asymptotic relationship

$$P_n(x;\alpha,\beta,\gamma:q)\sim\frac{x^n(\alpha q/x;q)_\infty\,(\gamma q/x;q)_\infty}{(\gamma q;q)_\infty\,(\alpha q;q)_\infty} \tag{1.17}$$

as $n\to\infty$, for fixed $x\neq 0$, αq^{m+1}, γq^{m+1}, $m=0,1,\ldots$, uniformly for x, α, β, and γ in compact sets. This describes the asymptotic behavior of the polynomials off the spectrum. We shall also prove

$$P_n(\alpha q^{m+1};\alpha,\beta,\gamma:q)\sim\frac{(\alpha\beta q^{m+1}/\gamma;q)_\infty\,\alpha^m(-1)^n}{(\gamma q;q)_\infty\,(\alpha q;q)_m}\gamma^{n-m}q^{m^2+\frac{1}{2}n(n+1)-mn} \tag{1.18}$$

as $n\to\infty$, uniformly for α, β, and γ in compact sets.

The remaining spectral points are $x=\gamma q^{m+1}$, $m=0,1,\ldots$. The asymptotic behavior at these points follows from (1.18) and the symmetry property $P_n(x;\gamma,\beta,\alpha:q)=P_n(x;\alpha,\gamma\beta/\alpha,\gamma:q)$, see (1.14).

Finally, Section 5 contains remarks on the relationship between asymptotic formulas of orthogonal polynomials $\{p_n(z)\}$ and the support of the measure that the p_n's are orthogonal with respect to. We also point out how the methods of the present work can provide asymptotic formulas for slightly more general polynomials.

2. THE LITTLE q-JACOBI POLYNOMIALS

Proof of (1.5). In the sum defining $\Phi_n^{\alpha,\beta}(x)$ (see (1.1) and (1.3)) replace the summation index k by $n-k$ and apply

$$(q^{-n};q)_{n-k}=(-1)^{n-k}\,q^{(k-n)(n+k+1)/2}(q;q)_n/(q;q)_k \tag{2.1}$$

to obtain

$$(-x)^{-n}\,\frac{(\alpha\beta q;q)_n}{(q;q)_n}\,q^{n(n-1)/2}\Phi_n^{\alpha,\beta}(x)=\sum_{k=0}^{n}\frac{(\alpha\beta q;q)_{2n-k}\,(-x)^{-k}\,q^{k(k-1)/2}}{(q;q)_{n-k}\,(\alpha q;q)_{n-k}\,(q;q)_k}. \tag{2.2}$$

If we can justify interchanging the summation and limiting processes we can use Euler's formula (see (6), p. 66 in [10])

$$\sum_{k=0}^{\infty}q^{k(k-1)/2}(-x)^k/(q;q)_k=(x;q)_\infty \tag{2.3}$$

to get

$$\lim_{n\to\infty}(-x)^{-n}\,q^{n(n-1)/2}\,\Phi_n^{\alpha,\beta}(x)=(x^{-1};q)_\infty/(\alpha q;q)_\infty. \tag{2.4}$$

Justifying the limit process is just an application of Tannery's theorem [11, p. 316]. Tannery's teorem is the discrete version of the Lebesgue dominated convergence theorem. Since $\{(\alpha\beta q;q)_n\}_{n=0}^{\infty}$ and $\{1/(q;q)_n\,(\alpha q;q)_n\}_{n=0}^{\infty}$ are both convergent sequences, the numbers $(\alpha\beta q;q)_{2n-k}/(q;q)_{n-k}\,(\alpha q;q)_{n-k}$, $0\leqslant k\leqslant n\leqslant\infty$, are uniformly bounded. Also, the series in (2.3) converges absolutely, so Tannery's theorem applies. Since all the bounds and convergence results used in the proof are uniform for x, α, and β in compact sets, so is (2.4). This proves formula (1.5).

Other proofs of (1.5) come from applying Tannery's theorem to the series in (2.7) below or from applying Darboux's method (see Section 3 for example) to the generating function (1.4).

Proof of (1.4). The transformation

$${}_2\phi_1\left(\begin{matrix}A,B;q,z\\C\end{matrix}\right)=\frac{(Az;q)_\infty}{(z;q)_\infty}\sum_{k=0}^{\infty}\frac{(A;q)_k\,(C/B;q)_k\,q^{k(k-1)/2}(-Bz)^k}{(C;q)_k\,(q;q)_k\,(Az;q)_k} \tag{2.5}$$

is the q-analogue of the Pfaff–Kummer transformation and was proven by Andrews [4] as a limiting case of an earlier identity of N. Hall connecting two ${}_3\phi_2$'s. When we apply (2.5) to the series defining $\Phi_n^{\alpha,\beta}(x)$, and manipulate the q-factorials using

$$(\lambda q^{-n};q)_k=(-\lambda)^k\, q^{k(k-1)/2-nk}(q/\lambda;q)_n/(q/\lambda;q)_{n-k}, \tag{2.6}$$

we find that

$$\frac{q^{n(n-1)/2}(-x)^{-n}}{(q;q)_n\,(\beta q;q)_n}\,\Phi_n^{\alpha,\beta}(x)=\sum_{k=0}^{n}\frac{(x^{-1};q)_{n-k}\,q^{k(k-1)}(\alpha q)^k}{(\beta q;q)_{n-k}\,(q;q)_{n-k}\,(\alpha q;q)_k\,(q;q)_k}. \tag{2.7}$$

Multiplying by t^n and summing from $n=0$ to ∞ immediately gives (1.4).

Proof of (1.6) *and* (1.7). Formula (1.7) is another version of (2.7), obtained by writing the sum in reverse order (replacing k by $n-k$ in each term) and then putting it in basic hypergeometric form by using (2.6) in the form

$$(\lambda;q)_{n-k}=(-\lambda)^{-k}\,q^{k(k+1)/2-nk}(\lambda;q)_n/(q^{-n+1}/\lambda;q)_k. \tag{2.8}$$

When $x=q^m$, $m=0,1,2,\ldots$, we see that the ${}_3\phi_2$ in (1.7) is a sum of $m+1$ terms, a polynomial in q^{-n}. As $n\to\infty$, the only asymptotically significant term is the $(m+1)$st term. This shows that

$$\Phi_n^{\alpha,\beta}(q^m)\sim\frac{(\beta q;q)_\infty}{(\alpha q;q)_\infty}\cdot\frac{q^{n(n+1)/2}(-\alpha q^m)^n\,q^{m(m-1)}\alpha^{-m}q^{-2nm}(q^{-m};q)_m\,q^m}{(\beta q;q)_m\,(q;q)_m}.$$

A little simplification yields (1.6).

3. The ${}_4\phi_3$ Polynomials

Proof of (1.9). Sears' transformation (see [5, 9, 16])

$$\begin{aligned}&{}_4\phi_3\begin{bmatrix}q^{-n},a,b,c;q,q\\ d,e,f\end{bmatrix}\\&\quad=\left(\frac{bc}{d}\right)^n\frac{(de/bc;q)_n\,(df/bc;q)_n}{(e;q)_n\,(f;q)_n}\,{}_4\phi_3\begin{bmatrix}q^{-n},a,d/b,d/c;q,q\\ d,de/bc,df/bc\end{bmatrix}\end{aligned} \tag{3.1}$$

(valid when $def=abcq^{-n+1}$) is the basic analogue of Whipple's ${}_4F_3$ transformation (1) [10, p. 56]. It contains as a limiting case the ${}_3\phi_2$ transformation from which Andrews proved (2.5). Applied to the series defining $W_n(x)$, it gives

$$W_n(x)=(ab;q)_n\,(cdzq^{n-1})^n\,(q^{-n+1}/dz;q)_n\,(q^{-n+1}/cz;q)_n$$
$$\times\,{}_4\phi_3\left[\begin{matrix}q^{-n},a/z,q^{-n+1}/cd,b/z;q,q\\ ab,q^{-n+1}/dz,q^{-n+1}/cz\end{matrix}\right].$$

The identities (2.6) and (2.8) allow us to write this as

$$\frac{W_n(x)}{(ab;q)_n\,(cd;q)_n\,(q;q)_n}$$
$$=\sum_{k=0}^{n}\frac{(a/z;q)_k\,(b/z;q)_k\,z^k}{(ab;q)_k\,(q;q)_k}\,\frac{(cz;q)_{n-k}\,(dz;q)_{n-k}\,z^{-n+k}}{(cd;q)_{n-k}\,(q;q)_{n-k}}. \tag{3.2}$$

Multiplying by t^n and summing over n leads to (1.9).

Proof of (1.11). We let n tend to ∞ in (3.2). The numbers $(cz;q)_{n-k}$ $(dz;q)_{n-k}/(cd;q)_{n-k}\,(q;q)_{n-k}$, $0\leqslant k\leqslant n\leqslant\infty$, are uniformly bounded, since the sequence $\{(cz;q)_n\,(dz;q)_n/(cd;q)_n\,(q;q)_n\}_{n=0}^{\infty}$ converges. Also the series

$$\sum_{k=0}^{n}\frac{(a/z;q)_k\,(b/z;q)_k\,z^{2k}}{(ab;q)_k\,(q;q)_k}$$

converges absolutely and has the sum $(az;q)_\infty\,(bz;q)_\infty/(ab;q)_\infty\,(z^2;q)_\infty$ by the q-analogue of Gauss' theorem [10, p. 68]. So Tannery's theorem gives

$$\frac{W_n(x)}{(ab;q)_n\,(cd;q)_n\,(q;q)_n}\sim z^{-n}\,\frac{(cz;q)_\infty\,(dz;q)_\infty}{(cd;q)_\infty\,(q;q)_\infty}\,\frac{(az;q)_\infty\,(bz;q)_\infty}{(ab;q)_\infty\,(z^2;q)_\infty},$$

which simplifies to (1.11). The result holds uniformly for a, b, c, d, and z in compact sets because the bound and convergence theorem used in the proof hold uniformly.

Another proof of (1.11) uses Darboux's method and is similar to the following proof.

Proof of (1.12). We need a slight generalization of a theorem of Darboux. If $f(t)=\sum_{n=0}^{\infty}f_n t^n$ and $g(t)=\sum_{n=0}^{\infty}g_n t^n$ are analytic in $\{|t|<r\}$ and $f(t)-g(t)$ is continuous in $\{|t|\leqslant r\}$, then $f_n=g_n+0(r^{-n})$. The theorem and its proof in [14, p. 310] are easily extended to show that if $f(t)$ and $g(t)$ depend on parameters $a_1,\dots,a_m$ and $f(t)-g(t)$ is a continuous function of $t, a_1, a_2,\dots, a_m$ for $a_1, a_2,\dots, a_m$ restricted to compact sets and $|t|\leqslant r$, then the conclusion holds uniformly with respect to the parameters.

We use the generating function on the right-hand side of (1.9) as $f(t)$. The nature of the singularities of a ${}_2\phi_1$ is revealed by the q-Pfaff–Kummer transformation (2.5). In fact, $f(t)$ has simple poles at $t=z^{\pm 1}$ and no other singularities in $\{|t|\leqslant|q|^{-1/2}\}$. Furthermore,

$$\begin{aligned}
&\lim_{t\to z}(1-t/z)f(t)\\
&= {}_2\phi_1\begin{pmatrix} a/z, b/z; q, z^2 \\ ab \end{pmatrix}\frac{(cz;q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(c/z;q)_k\, q^{k(k-1)/2}(-dz)^k}{(cd;q)_k\,(q;q)_k}\\
&= {}_2\phi_1\begin{pmatrix} a/z, b/z; q, z^2 \\ ab \end{pmatrix}\frac{(cz;q)_\infty}{(q;q)_\infty}\lim_{\omega\to\infty}{}_2\phi_1\begin{pmatrix} c/z, \omega; q, dz/\omega \\ cd \end{pmatrix}\\
&= \frac{(az;q)_\infty\,(bz;q)_\infty\,(cz;q)_\infty}{(z^2;q)_\infty\,(ab;q)_\infty\,(q;q)_\infty}\lim_{\omega\to\infty}\frac{(dz;q)_\infty\,(cd/\omega;q)_\infty}{(cd;q)_\infty\,(dz/\omega;q)_\infty}\\
&= A(z)\{(ab;q)_\infty\,(cd;q)_\infty\,(q;q)_\infty\}^{-1},
\end{aligned}$$

where the ${}_2\phi_1$'s were evaluated by the q-analogue of Gauss' summation theorem (see [10, p. 68] or [17, (3.3.2.5), p. 97]). By symmetry,

$$\lim_{t\to z^{-1}}(1-tz)f(t) = A(z^{-1})/(ab;q)_\infty\,(cd;q)_\infty\,(q;q)_\infty.$$

Thus, a suitable comparison function is

$$g(t) = \frac{A(z)(1-t/z)^{-1}+A(z^{-1})(1-tz)^{-1}}{(ab;q)_\infty\,(cd;q)_\infty\,(q;q)_\infty},$$

and Darboux's method yields (1.12).

4. The Big q-Jacobi Polynomials

One way to prove (1.15) is to set $f=(abc/de)\,q^{1-n}$ and let $a\to 0$ in Sears' transformation (3.1). This leads to

$$P_n(x;\alpha,\beta,\gamma:q) = \frac{(\beta x q^n)^n\,(\gamma q^{1-n}/\beta x;q)_n}{(\gamma q;q)_n}\,{}_3\phi_2\left[\begin{matrix} q^{-n}, \dfrac{q^{-n}}{\beta}, \dfrac{\alpha q}{x}; q, q \\ \alpha q, \dfrac{\gamma q^{1-n}}{\beta x} \end{matrix}\right], \tag{4.1}$$

which reduces to (1.15) after some manipulations.

Proof of (1.16). Let k be the summation index in the ${}_3\phi_2$ function in (1.15). Clearly $n\geqslant k\geqslant 0$. Replace k by $n-k$ and use (2.8) to get

$$\begin{aligned}
P_n(x;\alpha,\beta,\gamma:q) &= \frac{(q;q)_n\,(q\beta;q)_n}{(q\gamma;q)_n}\\
&\times\sum_{k=0}^{n}\frac{(\alpha q/x;q)_{n-k}\,(\beta x/\gamma;q)_k\,x^{n-k}(-\gamma)^k}{(\alpha q;q)_{n-(k}\,(q;q)_{n-k}\,(q;q)_k\,(q\beta;q)_k}\,q^{k(k+1)/2}.
\end{aligned}\tag{4.2}$$

Now multiply both sides of (4.2) by $t^n(q\gamma; q)_n \{(q;q)_n (q\beta; q)_n\}^{-1}$ and sum over $n \geqslant 0$. The interchange of the sums over n and k yields (1.16).

Proof of (1.17). The argument is similar to the one used to prove (1.6). Tannery's theorem implies

$$\begin{aligned}
&\lim_{n\to\infty} \frac{x^{-n}(q\gamma; q)_n}{(q\beta; q)_n} P_n(x; \alpha, \beta, \gamma : q) \\
&= \frac{(\alpha q/x; q)_\infty}{(\alpha q; q)_\infty} \sum_{k=0}^{\infty} \frac{(\beta x/\gamma; q)_k (-\gamma q/x)^k}{(q;q)_k (q\beta; q)_k} q^{k(k-1)/2} \\
&= \frac{(\alpha q/x; q)_\infty}{(\alpha q; q)_\infty} \lim_{\omega\to 0} {}_2\phi_1 \left[\begin{matrix}\beta x/\gamma, \gamma q/x\omega; q, \omega \\ q\beta\end{matrix}\right] \\
&= \frac{(\alpha q/x; q)_\infty (\gamma q/x; q)_\infty}{(\beta q; q)_\infty (\alpha q; q)_\infty}
\end{aligned}$$

by the q-analogue of Gauss' theorem [10, p. 68 or 17, p. 97]. This proves (1.17).

Finally we come to (1.18).

Proof of (1.18). When $x = \alpha q^{m+1}$, (1.15) reduces to

$$\begin{aligned}
P_n(\alpha q^{m+1}; \alpha, \beta, \gamma : q) &= \frac{(-\gamma)^n q^{n(n+1)/2}}{(\gamma q; q)_n} (\alpha\beta q^{m+1}/\gamma; q)_n \\
&\quad \times {}_3\phi_2 \left[\begin{matrix} q^{-m}, q^{-n}, q^{-n}/\beta; q, q \\ \alpha q, \gamma q^{-n-m}/\beta\alpha \end{matrix}\right].
\end{aligned} \tag{4.3}$$

The summation index, say k, in the above ${}_3\phi_2$ varies between 0 and m when $n \geqslant m$. Hence for sufficiently large n, the interchange of the finite summation and limiting processes is justified. The dominant term in the sum is the term $k = m$. Therefore we have, by (4.3),

$$\begin{aligned}
P_n(q^{m+1}; \alpha, \beta, \gamma : q) &\sim \frac{(-\gamma)^n q^{n(n+1)/2} (\alpha\beta q^{m+1}/\gamma; q)_\infty (q^{-m}; q)_m}{(\gamma q; q)_\infty (\alpha q; q)_m (q;q)_m} \\
&\quad \times \left(-\frac{\alpha}{\gamma}\right)^m q^{m(3m+1-2n)/2},
\end{aligned}$$

which reduces to (1.18) after some simplification.

5. Concluding Remarks

The asymptotic behavior of orthogonal polynomials on and off the interval of orthogonality is closely related to the measure that the polynomials are orthogonal with respect to; see Askey and Ismail [7], Nevai [13], and Szegö [19] for details and references. The Szegö class is the class of polynomials $\{P_n(z)\}$ that are orthogonal on $(-1, 1)$ with respect a measure μ with $d\mu/dx = w(x)$, $w > 0$ on $(-1, 1)$ and $\int_{-1}^{1} |\log w(x)|\, dx/\sqrt{1-x^2} < \infty$. The Szegö theory determines the asymptotic behavior of $P_n(z)$ as $n \to \infty$ and z fixed, $z \neq \pm 1$ in terms of w; see Szegö [19, Chap. 12] for details. The little and big q-Jacobi polynomials represent the other extreme, that is, discrete orthogonal polynomials. The rule of thumb seems to be that the point masses of the measure are at the points z that make the coefficient of the main term in the asymptotic expansion of $P_n(z)$ vanish. For example, (1.5) suggests that the support of discrete component of the measure corresponding to the little q-Jacobi polynomials coincides with the points $x = q^k$, $k = 0, 1, \ldots$. Furthermore, the absence of oscillatory terms in (1.5) suggests that the measure is purely discrete. These guesses are indeed correct and can be proved from general theorems when

$$-1 < q < 1, \qquad 0 < \alpha q < 1, \qquad \beta q < 1. \tag{5.1}$$

The little q-Jacobi polynomials satisfy the orthogonality relation

$$\sum_{j=0}^{\infty} \frac{\alpha^j q^j (q^{j+1}; q)_\infty}{(\beta q^{j+1}; q)_\infty} \Phi_n^{\alpha,\beta}(q^j)\, \Phi_m^{\alpha,\beta}(q^j)$$
$$= \frac{\alpha^n q^n (\alpha\beta q^{n+1}; q)_\infty\, (q; q)_n\, \delta_{m,n}}{(\beta q^{n+1}; q)_\infty\, (\alpha q; q)_\infty\, (\alpha q; q)_n\, (1 - \alpha\beta q^{2n+1})}. \tag{5.2}$$

The condition (5.1) guarantees the positivity of the discrete measure in (5.2). On the other hand (5.2) remains valid under the less restrictive conditions

$$-1 < q < 1, \qquad |\alpha q| < 1, \qquad \beta \neq q^{-k}, \quad k = 1, 2, \ldots. \tag{5.3}$$

When (5.3) holds but (5.1) is violated the point masses in (5.2) change sign. So guessing the support of the measure from the asymptotics seems to work even when the measure is not necessarily positive. The reason it works when the measure is positive is that when the moment problem is determined, that is, the positive measure is unique, the Christoffel function $\rho(z) = \sum_0^\infty |P_n(z)|^2$ diverges off the discrete spectrum and converges on it, where $P_n(z)$ are the orthonormal polynomials. In fact if $\rho(x_0) < \infty$, then the mass at x_0 is $1/\rho(x_0)$. There are no such results known for signed measures because such measures are not unique. Formula (1.17) similarly suggests that the big q-

Jacobi polynomials are orthogonal with respect to a purely discrete measure supported at the points αq^{m+1} and γq^{m+1}, which is also true (see Andrews and Askey [5]).

We now analyze (1.11) and (1.13) in view of the above remarks. Clearly (1.11) and (1.13) suggest that the support of the absolutely continuous component of the corresponding measure is $[-1, 1]$. Moreover, the coefficient of the main term in the asymptotic expansion of $W_n(x)$ vanishes if and only if az, bz, cz, or dz is q^{-k}, $k = 0, 1, \ldots$. We first consider solutions to $az = q^{-k}$. If $|a| < 1$, az cannot take any of the values q^{-k}. On the other hand if $a > 1$, $az = q^{-k}$ will have only finitely many solutions (x's) because z is the root of $z^2 - 2xz + 1$ that has absolute value less than unity. Similarly, $bz = q^{-k}$, $cz = q^{-k}$, or $dz = q^{-k}$ have at most finitely many solutions. The ${}_4\phi_3$ polynomials belong to the Szegö class when $|a| \leqslant 1$, $|b| \leqslant 1$, $|c| \leqslant 1$, $|d| \leqslant 1$, so (1.11) and (1.12) also follow, in this case, from Theorems 12.1.2 and 12.1.4 in Szegö [19].

It is worth mentioning that the way we proved (1.5) can be used to determine the asymptotic behavior of the more general polynomials

$$\Theta_n(x) = {}_{r+1}\phi_r \left[\begin{matrix} q^{-n}, a_1 q^n, \ldots, a_s q^n, a_{s+1}, \ldots, a_r; q, qx \\ b_1 q^n, \ldots, b_t q^n, b_{t+1}, \ldots, b_r \end{matrix} \right].$$

Following the same steps of the proof of (1.5) we obtain

$$\lim_{n \to \infty} (-x)^{-n} q^{n(n-1)/2} \Theta_n(x) = \left[\frac{(a_{s+1}; q)_\infty \cdots (a_r; q)_\infty}{(b_{t+1}; q)_\infty \cdots (b_r; q)_\infty} \right] (x^{-1}; q)_\infty .$$

It is easy to generalize (1.11) and (1.17) in a similar fashion.

The idea of using Tannery's theorem to compute the main terms in the asymptotic expansion for orthogonal polynomials has been used by Watson [20], Ismail [12], and Al-Salam and Ismail [3] in the cases of Lommel, q-Lommel, and the $U_n(x; a, b)$'s, respectively. The U_n's are generated by $U_0(x; a, b) = 1$, $U_1(x; a, b) = x(1 + a)$, $U_{n+1}(x; a, b) = x(1 + aq^n) U_n(x; a, b) - bq^{n-1} U_{n-1}(x; a, b)$, $n > 0$.

Acknowledgment

The authors are indebted to Richard Askey for several improvements in this paper, and much more so for the lasting influence of his tutelage.

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